EQUILIBRIUM THEORY AND STABILITY OF A HIGH-CURRENT, PINCHED, VARYING-CURRENT DISCHARGE

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The characteristics of a quasistationary state of a varying-current discharge are considered when the discharge current can be assumed to be high-frequency relative to hydrodynamic processes and constant relative to electrodynamic processes. It is shown that the quasistationary state of such a discharge is described by the same relations as the equilibrium state of a constant current discharge, but with physical quantities replaced by corresponding effective values. The discharge considered seems as unstable as a constant current discharge.

The equilibrium and stability theory of high-current, pinched discharges in a dense plasma, developed in recent years for the case where radiation plays a dominant role in the energetics balance [1-5], is applicable for a quasistationary discharge. This implies that the characteristic variation time of the discharge current should be much longer than the characteristic stability time of hydrodynamic equilibrium, i.e.,

$$\frac{T}{4} \sim \left(\frac{d \ln I}{dt}\right)^{-1} \gg \frac{a}{v_S} \tag{0.1}$$

where I is the strength of the current discharge, T is the period, a is the characteristic size of the discharge channel, and v_S is the isometric sound velocity in the plasma.

In a number of papers [5-7] devoted to experimental verification of this theory, condition (0.1) appears not to be always satisfied. The main such circumstance is that condition (0.1), as a rule, can be satisfied only in devices with sufficiently large periods, as in [8], for example. At the same time, however, it is first, hard to achieve a large discharge current, and, secondly, in the flow of the main discharge phase various instabilities are developed [5, 9, 10]. Therefore comparison of experiment and theory [5-7] is performed only for a comparatively short time interval close to the first maximum of the discharge current, at which the current channel is tightened by the intrinsic magnetic field and is sufficiently stable, while at the same time the whole process is continuous for several periods. On the other hand, both in an oscillating vacuum discharge and in an atmospheric discharge with a small period [7, 11], when inequality (0.1) is destroyed the time behavior of the discharge radius can be described by the usual expression, obtained for a quasistationary discharge in which one should only replace $I(t) = I_{ef}$, where $I_{ef} = I_0/\sqrt{2}$ is the effective value of the current strength. As to the time behavior of the discharge temperature, experimentally it seems to be strongly oscillating. This fact and also the assumption [11] that a quickly oscillating discharge can become more stable due to a dynamic stabilization process renders the problem of constructing a theory of pinched discharges of a varying current, to which the present paper is also devoted, quite timely.

1. Statement of the Problem and Starting Equations. Following the remarks above, we define a varying-current discharge as a discharge in which the following inequalities are satisfied:

$$\frac{v_S}{a} \ll \omega_0 \ll \frac{c^2}{4\pi\sigma_0 a^2} \tag{1.1}$$

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where ω_0 is the current vibrational frequency in the external chain, and σ_0 is the conductivity of the plasma, which is assumed to be completely ionized. The inequalities express the fact that the discharge current can be assumed high-frequency relative to hydrodynamic processes and constant relative to electrodynamic processes (magnetic field diffusion in the plasma, the absence of a scanning current).

To study equilibrium and stability we use the magnetohydrodynamic model including radiation in the form [1-4]

$$
\begin{gathered}
\operatorname{div} \mathbf{B}=0, \quad \operatorname{rot} \mathbf{B}=\frac{4\pi}{c} \mathbf{j}=\frac{4\pi}{c} \sigma\left\{\mathbf{E}+\frac{1}{c}[\mathbf{v}\mathbf{B}]\right\} \\
-c \operatorname{rot} \mathbf{E}=\frac{\partial \mathbf{B}}{\partial t}=\operatorname{rot}[\mathbf{v}\mathbf{B}]-\frac{c^2}{4\pi} \operatorname{rot}\left(\frac{1}{\sigma} \operatorname{rot} \mathbf{B}\right) \\
\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla) \mathbf{v}\right]=-\nabla P+\frac{1}{4\pi}[\operatorname{rot} \mathbf{B} \cdot \mathbf{B}] \\
\frac{\partial \rho}{\partial t}+\operatorname{div} \rho \mathbf{v}=0 \\
\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}+\rho \varepsilon\right)+\operatorname{div}\left\{\rho \mathbf{v}\left(\frac{v^2}{2}+\varepsilon+\frac{P}{\rho}\right)\right\}=\mathbf{j}\mathbf{E}-\operatorname{div} \mathrm{S} \\
P=\frac{(1+Z) k T}{M} \rho=v_S{}^2 \rho, \quad S_i=\frac{(1+Z) k}{M} \ln \frac{T^{3/2}}{\rho} \\
\sigma=\alpha Z^{-1} T^{3/2}, \qquad \alpha=4 \cdot 10^7
\end{gathered}
\tag{1.2}
$$

where E and B are the electric and magnetic fields, ρ is the plasma density, v is the velocity, P is the pressure, ε is the energy density, S_i is the ideal gas entropy, and Z and M are the ionic charge and mass. We neglect viscosity and electronic thermal conductivity. Radiation is taken into account by the radiation current S, determined by simultaneous solution of the system (1.2) and the radiation transfer equations.

We consider the case of an optically opaque plasma where $a \gg l_R$; here l_R is the scattering mean free path of light quanta, and the radiative thermal conductivity approximation [1] is valid for the current S

$$
\mathrm{S}=-{}^{16}/_3 \sigma^* T^3 l_R(\rho, T) \nabla T. \tag{1.3}
$$

We also consider the case of an optically transparent plasma, where $a \ll l_R$ and, according to [2],

$$
q_S=\operatorname{div} \mathrm{S}=\gamma T^{1/2} N^2 Z^3. \tag{1.4}
$$

Here σ^* is the Stefan–Boltzmann constant and the quantities l_R and γ depend on the specific light absorption mechanism in the plasma. The corresponding values were given in [1, 2].

In this paper we investigate the quasistationary discharge state and its stability. We define the concept of a quasiequilibrium state, making the following observations. Due to the short hydrodynamic time compared to the varying current period in the external chain the magnetic field can follow changes in the current discharge, i.e., if $I(t) = I_0 \sin \omega_0 t$, then

$$
\mathbf{j}_e=\mathbf{j}_{e0} \sin \omega_0 t, \qquad \mathbf{B}_e=\mathbf{B}_{e0} \sin \omega_0 t\,. \tag{1.5}
$$

The energy balance in the discharge is determined by ohmic heat separation and heat transfer from the plasma due to radiation

$$
j_e^2 / \sigma_e=q_S(\rho_e, T_e) \tag{1.6}
$$

The time dependence of the discharge temperature is also determined similarly. Thus, the discharge temperature is also strongly oscillating.

The hydrodynamic plasma pressure and the magnetic pressure are also oscillating, so that the velocity also oscillates in the discharge process. However, since $\omega_0 \gg v_S/a$, the plasma density should remain practically constant, and, consequently, the velocity oscillations are so small that they can be neglected. Such a state will also be called quasistationary, and its characteristic quantities are denoted by the index e.

The assumption made concerning the variation of hydrodynamic quantities can be verified even more rigorously by a calculation. Indeed, we consider the solution of the equations of motion and of continuity following from the specific time dependence of the quantities $\mathbf{j}_e$, $\mathbf{B}_e$, and P_e, assuming them to be as oscillating as the current. We look for solutions of the equations in the form of a sum of terms smoothly varying in time and of small terms quickly oscillating with time, whose averages in a period $2\pi/\omega_0$ vanish

$$\rho = \rho_e + \rho_e', \quad \rho_e' \ll \rho_e, \quad \int_0^{2\pi} \rho_e' dt = 0$$
$$v = v_e + v_e', \quad v_e' \ll v_e, \quad \int_0^{2\pi} v_e' dt = 0. \tag{1.7}$$

Substituting (1.7) in the equations of motion and of continuity written in a cylindrical coordinate system, and retaining first-order terms in ρ_e' and v_e', we obtain a system of four equations, two equations for the smooth terms and two for the oscillating terms

$$\frac{\partial \rho_e}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_e v_e) = 0, \quad \rho_e\left(\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial r}\right) = 0$$
$$\frac{\partial \rho_e'}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_e v_e') + \frac{1}{r}\frac{\partial}{\partial r}(r\rho_e' v_e) = 0 \tag{1.8}$$
$$\rho_e \frac{\partial v_e'}{\partial t} + \rho_e' \frac{\partial v_e}{\partial t} + \rho_e v_e \frac{\partial v_e'}{\partial r} + \rho_e v_e' \frac{\partial v_e}{\partial r} +$$
$$+ \rho_e' v_e \frac{\partial v_e}{\partial r} = - v_S^2(t) \frac{\partial \rho_e}{\partial r} - v_S^2(t) \frac{\partial \rho_e'}{\partial r} - \frac{1}{c} j_e B_e .$$

It follows automatically from the second equation of this system that $v_e = 0$, and from the first that the equilibrium density ρ_e is constant in time. The radial distribution of ρ_e is found by time averaging of the last equation of system (1.8), whence follows the equality of the average hydrodynamic and magnetic pressures in the quasiequilibrium state

$$\partial \langle P_e \rangle / \partial r + \frac{1}{c} \langle j_e B_e \rangle = 0 .$$

Thus, the quasiequilibrium discharge state should be described by the system of equations

$$\frac{1}{r}\frac{\partial}{\partial r} r B_e = \frac{4\pi}{c} j_e = \frac{4\pi}{c} \sigma_e E_e, \quad \frac{\partial \langle P_e \rangle}{\partial r} + \frac{1}{c}\langle j_e B_e \rangle = 0$$
$$\frac{j_e^2}{\sigma_e} = q_S(\rho_e, T_e) = \frac{1}{r}\frac{\partial}{\partial r} r S(\rho_e, T_e) \tag{1.9}$$

applicable for studying direct and inverse discharges in both optically opaque and transparent plasmas.

2. Quasiequilibrium Discharge State. We first consider the quasiequilibrium discharge state for an optically opaque plasma. For the current S we then have expression (1.3), with whose inclusion the last equation of system (1.9), the energy balance equation, is written in the form

$$\frac{j_e^2}{\sigma_e} \pi r_0^2 = \sigma^* T_e^4 2\pi r_0 \tag{2.1}$$

where r_0 is the discharge radius. Equation (2.1) represents the equality of ohmic heat separation in the plasma bulk to blackbody radiation. It is easy to obtain from this relation the time dependence of the plasma temperature

$$T_e = T_{e0} \sin^{4/11} \omega_0 t . \tag{2.2}$$

Thus, in an optically opaque discharge, the temperature oscillates with time with the external field frequency. However, its variation close to the current maxima is significantly slower than that of a harmonic law.

We turn attention now to the following fact: the first two equations of system (1.9) can be reduced to the form

$$\frac{1}{r}\frac{\partial}{\partial r} r B_{ef} = \frac{4\pi}{c} j_{ef} = \frac{4\pi}{c} \sigma_{e0} E_{ef}$$
$$\frac{\partial P_{ef}}{\partial r} + \frac{1}{c} j_{ef} B_{ef} = 0 \tag{2.3}$$

by the substitutions

$$I_{ef} = \frac{I_{e0}}{\sqrt{2}}, \quad E_{ef} = \frac{E_{e0}}{\sqrt{2}}, \quad B_{ef} = \frac{B_{e0}}{\sqrt{2}}$$
$$P_{ef} = \langle P_e \rangle = \delta_1 P_{e0} = \frac{\Gamma(15/22)}{\sqrt{\pi}\,\Gamma(13/11)} P_{e0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{e0} \sin^{4/11} x dx . \tag{2.4}$$

The expressions for j_{ej}, E_{ef}, and B_{ef} have the simple physical meaning of effective values of varying current density and varying electrodynamic field intensity.

For a cylindrical discharge, the system of equations (2.3) should be supplemented by the boundary condition

$$B_{ef}\,|_{r=r_0} = \frac{2I_{ef}}{cr_0} \tag{2.5}$$

The system of equations (2.1) and (2.3) with the boundary condition (2.5), with the requirement of a bound solution at zero, coincides fully with the system of equations for a constant current [1, 3], and there is no need to solve it. The quasiequilibrium discharge state of a varying current will be described by the same relations as the equilibrium discharge state of a constant current, but replacing physical quantities by corresponding effective values. To sum up, we obtain for a quasiequilibrium state with a homogeneous temperature

$$B_{ef} = \sqrt{4\pi P_{ef}(0)}\,r/r_0, \quad P_{ef} = P_{ef}(0)(1 - r^2/r_0^2)$$
$$\rho_e = \rho_e(0)(1 - r^2/r_0^2), \quad r_0^2 = \frac{P_{ef}(0)\,c^2}{\pi j_{ef}^2}\,. \tag{2.6}$$

We can use all other relations obtained for an opaque discharge, in particular, expressions of its parameters in terms of the total current and particle number in the discharge N_n

$$T_{ef} = \delta_1 T_{e0} = \frac{I_{ef}^2}{2(1+Z)\,c^2 k N_n}\,. \tag{2.7}$$

We dwell shortly on the quasiequilibrium discharge in an optically transparent plasma. Taking into account the expression for the energy flow (1.4), the time dependence of temperature is obtained in this case from the energy balance equation

$$T_e = T_{e0}\,|\sin\omega_0 t\,|. \tag{2.8}$$

The temperature in a transparent discharge seems to oscillate, in agreement with the current oscillations.

In the case of a transparent discharge, the quasiequilibrium discharge state can be described by system (2.3), in which the definition of effective values of electromagnetic quantities is retained as previously and

$$P_{ef} = \langle P_e \rangle = \frac{P_{e0}}{2\pi}\int_{-\pi}^{\pi} |\sin x\,|\,dx = \frac{2}{\pi}P_{e0}. \tag{2.9}$$

Writing the boundary conditions for a transparent discharge in the form

$$rB_{ef}\,|_{r\to\infty} = \frac{2I_{ef}}{c}, \quad P_{ef}\,|_{r\to\infty} = 0 \tag{2.10}$$

and requiring a finite solution at zero, we arrive at the conclusion that the solutions for the effective quantities in a variable current discharge are expressed by the same equations as for a constant current discharge in a transparent plasma, but with replacing physical quantities by their effective values. For the cases of a retarding radiation mechanism and radiation by multiply ionized atoms, we obtain, following the results of [2, 3],

$$P_{ef} = \frac{P_{ef}(0)}{(1 + r^2/r_0^2)^2}, \quad B_{ef} = \sqrt{8\pi P_{ef}(0)}\,\frac{r/r_0}{1 + r^2/r_0^2}$$
$$r_0 = \sqrt{\frac{2}{\pi P_{ef}(0)}\,\frac{\beta_{ef} c Z}{\alpha E_{ef}}}, \quad P_{ef} = \beta_{ef} T_{e0}^{3/2} \tag{2.11}$$
$$\beta_{ef} = \frac{2}{\pi}\sqrt{\frac{2\alpha E_{ef}^2 k^2 (1+Z)^2}{\gamma Z^4}}\,.$$

Similarly we obtain corresponding equations for opaque and transparent discharges with reverse current.

3. Stability of a Quasiequilibrium State. The stability analysis of a quasiequilibrium state was performed by linearizing the system of equations (1.2) with respect to the small perturbations

$$\rho \to \rho_e + \rho, \; T \to T_e + T, \; P \to P_e + P, \; \mathbf{B} \to \mathbf{B}_e + \mathbf{B}, \; \mathbf{v}$$

by the normal wave method. Due to the time dependence of the quasiequilibrium quantities, the linearized system of equations is a system of differential equations with periodic coefficients of period $T = 2\pi/\omega_0$.

We consider an optically opaque discharge. In this case one can neglect the effect of vibrational temperature on the discharge stability process if the heat current due to radiative thermal conductivity S is stronger than the hydrodynamic heat current, which is automatically satisfied in a low conductivity plasma [1, 3]. This implies the absence of superheating instability in such a discharge. Since the time of development of power instabilities cannot be less than the hydrodynamic time r_0/v_S, one should restrict consideration to processes for which

$$\omega \lesssim \frac{v_S}{r_0} \ll \omega_0 \tag{3.1}$$

From the nature of the periodic force acting on the system one can, moreover, conclude that small perturbations around the quasiequilibrium values are almost periodic functions of period $2\pi/\omega_0$, i.e., they can be expanded in a Fourier series with slowly varying amplitudes

$$f = \sum_{n=-\infty} f_n \exp in\omega_0 t \qquad \left(\frac{1}{\omega_0}\left|\frac{\partial \ln f_n}{\partial t}\right| \ll 1\right) \tag{3.2}$$

Not all slowly varying amplitudes f_n, however, exist in the instability development process. In fact, only those f_n appearing in the equations of motion and of continuity, averaged over the current vibrational period, should be assumed to be nonvanishing.

Taking into account the symmetry of the problem, and the coordinate and time dependence of f_n, we write

$$f_n(t, \mathbf{r}) = f_n(\mathbf{r}) \exp(-i\omega t + im\varphi + ik_z z) \tag{3.3}$$

Perturbations with $m = 0$, $k_z \neq 0$, correspond usually to constrictions, and those with $m \neq 0$, $k_z \neq 0$ to kinks.

It follows from the stability analysis of an optically opaque constant current discharge that this discharge is subject to the most dangerous instability types of constriction and helical-kink instabilities. The increments of these instabilities are independent of conductivity. Therefore, the stability problem of an optically opaque discharge is reduced to studying the limiting case $\sigma_e \to 0$. After suitable calculations, we show that the power instabilities of a variable current discharge are described by the same system of equations as for a constant current discharge. One can immediately write the increment of instability development of a variable current discharge. For fundamental constriction modes, we have

$$\gamma = \left(2\sqrt{3}\,|k_z|\,\frac{v_{Sef}^2}{r_0}\right)^{1/2} \qquad (k_z r_0 < 1) \tag{3.4}$$

For long-wave helical instabilities the increment is $(k_z r_0)^{-1/2}$ times smaller. Here

$$v_{Sef}^2 = \frac{\Gamma(15/22)}{\sqrt{\pi}\,\Gamma(13/11)}\, v_{S0}{}^2 = \delta_1 v_{S0}{}^2 \tag{3.5}$$

where $\Gamma(x)$ is the Gamma function

External modes of long-wave constrictions have smaller increments, and, moreover, their increment drops with decreasing conductivity.

Consider now the stability of an optically transparent discharge. The stability study of an optically transparent constant current discharge shows that together with power instabilities in a low conductivity plasma there develops a high-frequency superheating instability with an increment

$$\gamma \sim c^2 / \sigma_0 a^2 \tag{3.6}$$

For a fast process the time variation of the current is meaningless, and it can be considered constant. Therefore the results concerning superheating instabilities of a constant current discharge remain valid also in the case of a varying current of frequency $\omega_0 \ll c^2/\sigma_0 a^2$.

A superheating can be stabilized with increasing temperature, since the existence condition of superheating $(v_S/a \ll c^2/\sigma a^2)$ is destroyed. The discharge stability is then determined by the evolution times of power instabilities, for which $\omega \lesssim v_S/a \ll \omega_0$. Such processes can be assumed slow compared to the period of a current in an external chain, and, consequently, the method of slowly varying amplitudes, developed in considering power instabilities of an opaque discharge, is applicable to them.

The absence of sharp boundaries of a transparent discharge leads to the vanishing of the fundamental constriction modes, independent of conductivity. However, besides the fundamental modes in a low conductivity plasma, there exist bulk oscillations of increment $\gamma \sim \sigma_e \to 0$. Calculations show that to obtain the increments of such oscillations it is sufficient to replace in the results of [2, 3]

$$v_S^2 \to v_{Sef}^2 = \frac{\sqrt{\pi}}{12} \frac{\Gamma(1/4)}{\Gamma(3/4)} v_{S_0}^2 \approx v_{S_0}^2 \quad (3.7)$$

Thus, the evolution increment of the most dangerous long-wave bulk oscillations for a simple Z-pinch equals

$$\gamma = \frac{k_z^2 r_0^2}{12(n+1/2)^3} \frac{\pi \sigma_{e0} v_{Sef}^2}{c^2} \quad (3.8)$$

We emphasize once more that a varying current discharge is subject to power instabilities of constriction and kink types, and in an optically transparent plasma, also to superheating. The increments obtained for optically opaque and transparent discharges differ from the previous ones (for a constant current discharge) by numerical factors of order unity; therefore a varying current discharge is as unstable as a constant current discharge.

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LITERATURE CITED

1. A. A. Rukhadze and S. A. Triger, "Equilibrium and stability of a high-current discharge in a dense plasma under conditions of radiative thermal conductivity," Zh. Prikl. Mekham. i Tekh. Fiz., No. 3, 11 (1968).
2. V. B. Rozanov, A. A. Rukhadze, and S. A. Triger, " Equilibrium and stability theory of high-current discharge in a dense optically transparent plasma," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 5, 18 (1968).
3. A. F. Aleksandrov, A. A. Rukhadze, and S. A. Triger, "Equilibrium and stability theory of powerful discharge in a dense plasma," 9th Intern. Conf. Phenomena Ionized Gases, Bucharest, s.a., 379(1969).
4. A. F. Aleksandrov, E. P. Kaminskaya, and A. A. Rukhadze, "Equilibrium and stability of a cylindric linear pinch with homogeneous temperature," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 1, 38 (1971).
5. A. F. Aleksandrov, V. V. Zosimov, A. A. Rukhadze, V. I. Savoskin, and I. B. Timofeev, "Theoretical and experimental studies of direct high-current discharges in vacuum," Preprint FIAN, No. 72 (1971).
6. A. F. Aleksandrov, V. V. Zosimov, A. A. Rukhadze, and V. I. Savoskin, "Temperature in boundary currents of a pinched linear opaque discharge," Kratkie Soobshchen. po Fiz., No. 6 (1970).
7. A. F. Aleksandrov, V. V. Zosimov, A. A. Rukhadze, V. I. Savoskin, and I. B. Timofeev, "High-current pinched discharges in an optically opaque plasma," 3rd All-Union Conf. Low Temperature Plasma Physics, Abstracts [in Russian], p. 176 (1971).
8. A. D. Klementov, G. V. Mikhailov, F. A. Nikolaev, V. B. Rozanov, and Yu. P. Sviridenko, "High-current pulse discharge in lithium," Teplofiz. Vys. Temp., 8, 736 (1970).
9. F. A. Nikolaev, V. B. Rozanov, and Yu. P. Sviridenko, "Instability of a high-current pulse discharge," Kratkie Soobshchen. po Fiz., No. 4, 59 (1971).
10. A. F. Aleksandrov, V. V. Zosimov, and I. B. Timofeev, "Power instability in a dense optically opaque plasma," Kratkie Soobshchen. po Fiz., No. 2, 25 (1972).
11. A. F. Aleksandrov, V. V. Zosimov, A. A. Rukhadze, and I. B. Timofeev, "A possible high-current pinched discharge mechanism in the atmosphere," Kratkie Soobshchen. po Fiz., No. 8, 72 (1970).